Lecture No. 28

Measure and Integration

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2/3/11

Step 1

$$E \in \mathcal{A} \otimes \mathcal{B}$$

$$\int_Y \left(\int_X \chi_E(x,y)\, d\mu \right) d\nu = (\mu \times \nu)(E) = \int \chi_E(x,y)\, d(\mu \times \nu)$$

$$= \int_X \left(\int_Y \chi_E(x,y)\, d\nu \right) d\mu$$

Claim holds for $f = \chi_E$, $E \in \mathcal{A} \otimes \mathcal{B}$

Step 2

Claim holds for $f = s$ a non-negative simple measurable function on $X \times Y$.

$$s = \sum_{i=1}^{n} a_i \chi_{E_i}, \quad E_i \in \mathcal{A} \otimes \mathcal{B}, \quad \bigsqcup_{i=1}^{n} E_i = X \times Y$$

Step 1 $\forall\, i = 1, 2, \ldots, n$

$$\sum \int_X \left(\int_Y a_i \chi_{E_i}(x,y)\, d\nu \right) d\mu = \sum_i \int a_i \chi_{E_i}\, d\mu \times \nu = \sum_i \int_Y \left(a_i \int_X \chi_{E_i}(x,y)\, d\mu \right) d\nu$$

Sum over i

$$\int_X \left(\int_Y \left(\sum_{i=1}^{n} a_i \chi_{E_i} \right) d\nu \right) d\mu$$

$$= \int_{X\times Y} \left(\sum_i a_i \chi_{E_i} \right) d\mu\times\nu$$

$$= \int_Y \left(\int_X \left(\sum_{i=1}^{n} a_i \chi_{E_i} \right) d\mu \right) d\nu$$

$$\int_X \left(\int_Y s(x,y)\, d\nu \right) d\mu = \int_{X\times Y} s(x,y)\, d\mu\times\nu$$

$$= \int_Y \left(\int_X s(x,y)\, d\mu(x) \right) d\nu$$

Step 3 $f : X \times Y \longrightarrow \mathbb{R}^*$, $f \geq 0$

f - $\mathcal{A} \otimes \mathcal{B}$ measurable.

$\Rightarrow$ $\exists$ a sequence $\{s_n\}_{n \geq 1}$ non-negative simple mble fns, $s_n(x,y) \uparrow f(x,y)$ $\forall (x,y) \in X \times Y$.

Step 2 $\forall$ s_n,

$$\underline{\underline{\int_X \left(\int_Y s_n(x,y)\, d\nu \right) d\mu}} = \int s_n(x,y)\, d(\mu \times \nu)(x,y) = \overline{}$$

$$\downarrow$$

$$\int f(x,y)\, d(\mu \times \nu)(x,y)$$

$$\int_X \left(\int_Y s_n(x,y)\, d\nu \right) d\mu$$

$$\longrightarrow \int_X \left(\int_Y f(x,y)\, d\nu \right) d\mu \ ?$$

<u>Note</u>

$$s_n(x,y) \uparrow f(x,y) \quad \forall\, (x,y)$$

Fix x,

$$y \longmapsto s_n(x,y) \quad \forall\, y \in Y$$

is an increasing sequence of non-negative fns

$y \longmapsto s_n(x,y)$ is a measurable function.

Let $s_n = \sum_{i=1}^{m_n} a_i^n \chi_{E_i^n}$

x fixed $y \longmapsto \chi_{E_i^n}(x,y)$ is mble.

For x fixed, $\{s_n(x,y)\}_{n\geq 1}$ is a sequence of non-negative $\mathcal{B}$-mble fns on Y. In fact it increases to $y \mapsto f(x,y)$.

By MCThm ($\forall x$ fixed)

$$\underbrace{\lim_{n\to\infty}\left(\int_Y s_n(x,y)\,d\nu(y)\right)} = \int_Y f(x,y)\,d\nu(y) \quad \circledast$$

$$\Longrightarrow \quad x \longmapsto \int_Y f(x,y)\,d\nu(y)$$

is (non-negative) measurable.

Another app. of MCT

$$\lim_{n\to\infty}\left(\int_X \left(\int_Y s_n(x,y)\,d\nu(y)\right) d\mu(x)\right.$$

$$= \int_X \left(\int_Y s_n(x,y)\, d\nu(y) \right) d\mu(x)$$

Also

$$\int_X \left(\int_Y s_n(x,y)\, d\nu(y) \right) d\mu(x) = \int_{X \times Y} s_n(x,y)\, d(\mu \times \nu)$$

$$\Rightarrow \lim_{n \to \infty} \left(\qquad \right) = \lim_{n \to \infty} \left(\qquad \right)$$

$$= \int_{X \times Y} f(x,y)\, d(\mu \times \nu)$$

$$\int_{X\times Y} f(x,y)\, d(\mu\times\nu)$$

$$= \lim_{n\to\infty} \Big(\int_X \Big(\int_Y s_n(x,y)\, d\nu(y) \Big) d\mu(x)$$

$$= \int_X \Big(\int_Y f(x,y)\, d\nu(y) \Big) d\mu(x)$$

$$= \int_Y \Big(\int_X f(x,y)\, d\nu(y) \Big) d\mu(x)$$

$\underline{f \in L_1(X \times Y)}$

$$\int_{X \times Y} f(x,y)\, d(\mu \times \nu)$$

$$\stackrel{?}{=} \int_X \left(\int_Y f(x,y)\, d\nu(y) \right) d\mu(x) \quad \Bigg| \quad \stackrel{?}{=} \int_Y \left(\int_X f(x,y)\, d\mu(x) \right) d\nu(y)$$

$$f \in L_1(X \times Y), \quad f = f^+ - f^-$$

$$\int_{X\times Y} f(x,y)\, d\mu\times\nu = \int_{X\times Y} f^+\, d\mu\times\nu - \int_{X\times Y} f^-\, d\mu\times\nu$$

$\Longrightarrow$ <u>F. Thm - I</u>

$$\int_{X\times Y} f^+(x,y)\, d\mu\times\nu = \int_X \left(\int_Y f^+(x,y)\, d\nu(y) \right) d\mu(x)$$

$$= \int_Y \left(\int_X f^+(x,y)\, d\mu(x) \right) d\nu(y)$$

$$\Longrightarrow \int_X \left(\int_Y f^+(x,y)\, d\nu(y) \right) d\mu(x) < +\infty$$

(Integral finite $\Rightarrow$ function finite a.e.)

$$\Longrightarrow \quad x \longmapsto \int_Y f^+(x,y)\, d\nu(y) \text{ is finite a.e.-}(x)$$

$$\Rightarrow \quad y \longmapsto f^+(x,y) \text{ is finite a.e.}$$

(integrable)

$\Rightarrow$ Similarly

$$x \longmapsto f^+(x,y) \text{ is finite a.e.}$$

and integrable.

$$\int_{X\times Y} f^+(x,y)\, d(\mu\times\nu) = \int_X \left(\int_Y f^+(x,y)\, d\nu \right) d\mu < +\infty$$

$$= \int_Y \left(\int_X f^+ d\nu \right) d\mu$$

$$\int_{X\times Y} f^-(x,y)\, d(\mu\times\nu) = \int_Y \left(\int_X f^-(x,y)\, d\nu \right) d\mu$$

$$< +\infty$$

Subtract $\Rightarrow$

$$\int_{X\times Y} f \, d\mu\times\nu = \int_Y \left(\int_X f(x,y)\, d\nu \right) d\mu.$$